# Epistemic *ATL* with Perfect Recall, Past and Strategy Contexts


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**Abstract.** We propose an extension to epistemic *ATL* with perfect recall, past, and distributed knowledge by strategy contexts and demonstrate the strong completeness of a Hilbert-style proof system for its $(.\mathsf{U}.)$-free subset.


## Introduction

*Alternating time temporal logic* (*ATL*, [2,3]) was introduced as a reasoning tool for the analysis of strategic abilities of coalitions in extensive multiplayer games with temporal winning conditions. Systems of *ATL* in the literature vary on their restrictions on the players' information on the game state, which may be either *complete* or *incomplete* (*imperfect*), and the players' ability to keep full record of the past, which is known as *perfect recall* [11,17].

The informal reading of the basic game-theoretic (cooperation) construct $\langle\langle\Gamma\rangle\rangle\varphi$ of *ATL* is *the members of coalition* $\Gamma$ *can cooperate to enforce temporal condition* $\varphi$ *regardless of the actions of the rest of the players*. Every player is either committed to the objective $\varphi$, or free to obstruct it. This restiction is overcome in *Strategy Logic* (*SL*, [6]), where propositional *LTL* language is combined with a predicate language interpreted over a domain of strategies to enable flexible quantification over strategies. *LTL* formulas are evaluated at the unique paths which are determined by dedicated parameter lists of strategies. For instance, assuming just two players 1 and 2, $\langle\langle 1\rangle\rangle(p\mathsf{U}q)$ translates into the *SL* formula $\exists x\forall y(p\mathsf{U}q)(x,y)$, where $(x,y)$ indicates evaluating $(p\mathsf{U}q)$ at the path determined by 1 and 2 following strategies $x$ and $y$, respectively. This translation is not invertible in general and *ATL* is not *expressively complete* wrt *SL*. Some practically interesting properties which cannot be written in *ATL* for this reason are given in [5]. To enable the expression of such properties, *ATL* was extended by *strategy contexts* in various ways [20,5,15,21]. Strategy contexts are assignments of strategies to some of the players which the rest of the players can safely assume to be followed. All of the works [20,15,21] are about strategy contexts in *ATL* with complete information. To facilitate reasoning about games with incomplete information, *ATL* was extended with epistemic operators [18,11]. Such combinations can be viewed as extending temporal logics of


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knowledge (cf. e.g [7]) in the way *ATL* extends computational tree logic *CTL*. A study of the system of epistemic linear- and branching-time *temporal* logics (without the game-theoretic modalities) which arise from the various possible choices can be found in [19,10].

In this work we embark on the study of an extension of epistemic *ATL* with perfect recall and past by strategy contexts. Our extension to the language of *ATL* is different from those in [15,21] but brings the same expressive power for the case of complete information. The language extension we chose has facilitated upgrading our axiomatic system for epistemic *ATL* with perfect recall from [9] to include strategy contexts by making only the obvious changes. Following [9], the semantics in this paper is based on the variant from [13,14] of *interpreted systems*, which are known from the study of knowledge-based programs [7]. The main result in the paper is the completeness of our proof system for the "basic" subset of epistemic *ATL* with past, perfect recall and strategy contexts. This subset excludes the iterative constructs $\langle\langle\Gamma\rangle\rangle(.\mathsf{U}.)$, $\langle\langle\Gamma\rangle\rangle\Diamond$ and $\langle\langle\Gamma\rangle\rangle\Box$, but includes the past operators $\ominus$ and $(.\mathsf{S}.)$, and following [9] again, the operator $\mathsf{D}_\Gamma$ of *distributed* knowledge. The future subset of the system can be viewed as an extension of Coalition Logic [16] as well. The system is compact. This enabled us to prove *strong* completeness, i.e., that an arbitrary consistent set of formulas is also satisfiable. The proof system includes axioms for temporal logic (cf. e.g. [12]), epistemic modal logic with distributed knowledge (cf. e.g. [7]), appropriately revised *ATL*-specific axioms and rules from the axiomatization of *ATL* with complete information in [8] and from the extension of *ATL* by strategy contexts proposed in [20], and some axioms from our previous work [9].

**Structure of the Paper.** After preliminaries on interpreted systems we introduce our logic. We briefly review the related logics from [20,15,21] and give a satisfaction preserving translation between our proposed logic and that from [15]. In the subsequent sections we present our proof system for the basic subset of the logic and demonstrate its completeness.

## 1 Preliminaries

In this paper we define $ATL_{iR}^{DPC}$ on *interpreted systems*. An *interpreted system* is defined with respect to some given finite set $\Sigma = \{1, \dots, N\}$ of *players*, and a set of *propositional variables* (*atomic propositions*) $AP$. There is also an *environment* $e \notin \Sigma$. In the sequel we write $\Sigma_e$ for $\Sigma \cup \{e\}$.

**Definition 1 (interpreted systems).** *An* interpreted system *for $\Sigma$ and $AP$ is a tuple of the form $\langle\langle L_i : i \in \Sigma_e\rangle, I, \langle Act_i : i \in \Sigma_e\rangle, t, V\rangle$ where:*

$L_i$, $i \in \Sigma_e$, *are nonempty sets of* local states*;* $L_\Gamma$ *stands for* $\prod\limits_{i\in\Gamma} L_i$, $\Gamma \subseteq \Sigma_e$*;*

$I \subseteq L_{\Sigma_e}$ *is a nonempty set of* initial global states*;*

$Act_i$, $i \in \Sigma_e$, *are nonempty sets of* actions*;* $Act_\Gamma$ *stands for* $\prod\limits_{i\in\Gamma} Act_i$*;*

$t : L_{\Sigma_e} \times Act_{\Sigma_e} \to L_{\Sigma_e}$ *is a* transition *function;*

$V \subseteq L_{\Sigma_e} \times AP$ *is a valuation of the atomic propositions.*

*The elements of* $L_{\Sigma_e}$ *are called* global states*. For every* $i \in \Sigma_e$ *and* $l', l'' \in L_{\Sigma_e}$ *such that* $l'_i = l''_i$ *and* $l'_e = l''_e$ *the function* $t$ *is required to satisfy* $(t(l', a))_i = (t(l'', a))_i$.

In the literature, interpreted systems also have a *protocol* $P_i : L_i \to \mathcal{P}(Act_i)$ for every $i \in \Sigma_e$. $P_i(l)$ is the set of actions which are available to $i$ at local state $l$. We assume the same sets of actions to be available to agents at all states for the sake of simplicity. For the rest of the paper in our working definitions we assume the considered interpreted system $IS$ to be clear from the context and its components to be named as above.

**Definition 2 (global runs).** *Given an* $n \leq \omega$, $r = l^0 a^0 l^1 a^1 \ldots \in L_{\Sigma_e}(Act_{\Sigma_e} L_{\Sigma_e})^n$ *is a* run of length $|r| = n$, *if* $l^0 \in I$ *and* $l^{j+1} = t(l^j, a^j)$ *for all* $j < n$. *We denote the set of all runs of length* $n$ *by* $R^n(IS)$. *We denote* $\bigcup\limits_{k<n} R^k(IS)$ *and* $\bigcup\limits_{k \leq n} R^n(IS)$ *by* $R^{<n}(IS)$ *and* $R^{\leq n}(IS)$, *respectively. We write* $R^{fin}(IS)$ *and* $R(IS)$ *for* $R^{<\omega}(IS)$ *and* $R^{\leq \omega}(IS)$, *respectively.*

*Given* $m, k < \omega$ *such that* $m \leq k \leq |r|$, *we write* $r[m..k]$ *for* $l^m a^m \ldots a^{k-1} l^k$. *We write* $R[m..k]$ *for* $\{r[m..k] : r \in R\}$ *in case the lengths of the runs in* $R \subset R(IS)$ *are at least* $k$.

Runs of length $n < \omega$ are indeed sequences of $2n + 1$ states and actions.

**Definition 3 (local states, local runs and indiscernibility of runs).** *Given an* $l \in L$ *and* $\Gamma \subseteq \Sigma_e$, *we write* $l_\Gamma$ *for* $\langle l_i : i \in \Gamma \rangle$; $a_\Gamma \in Act_\Gamma$ *is defined similarly for* $a \in Act_{\Sigma_e}$, *and indeed for* $a \in Act_\Delta$ *with arbitrary* $\Delta$ *such that* $\Gamma \subseteq \Delta \subseteq \Sigma_e$. *Sometimes we write* $l_\Gamma$ $(a_\Gamma)$ *just in order to emphasize that the index set of* $l$ $(a)$ *is* $\Gamma$. *Given* $r = l^0 a^0 \ldots \in R(IS)$, *we write* $r_\Gamma = l^0_\Gamma a^0_\Gamma \ldots$ *for the corresponding* local run *of* $\Gamma$. *Given* $r', r'' \in R(IS)$ *and* $n \leq |r'|, |r''|$, *we write* $r' \sim^n_\Gamma r''$ *if* $r'_\Gamma[0..n] = r''_\Gamma[0..n]$ *and* $r' \sim_\Gamma r''$ *for the conjunction of* $r' \sim^{|r'|}_\Gamma r''$ *and* $|r'| = |r''|$.

Obviously $\sim^n_\Gamma$ and $\sim_\Gamma$ are equivalence relations on $R(IS)$. We denote $\{r' \in R(IS) : r' \sim_\Gamma r\}$ by $[r]_\Gamma$. Sequences of the form $r_\emptyset$ consist of $\langle\rangle$s and $[r]_\emptyset$ is the class of all runs of length $|r|$.

**Definition 4 (joins of vectors of actions).** *Given two vectors* $a_i = \langle a_{i,j} : j \in \Gamma_i \rangle$, $i = 1, 2$, *such that* $\Gamma_1, \Gamma_2 \subseteq \Sigma_e$ *and* $\Gamma_1 \cap \Gamma_2 = \emptyset$, *we write* $a_1 \cup a_2$ *for the vector indexed by* $\Gamma_1 \cup \Gamma_2$ *with action* $(a_1 \cup a_2)_j$ *being either* $a_{1,j}$ *or* $a_{2,j}$, *depending on whether* $j \in \Gamma_1$ *or* $j \in \Gamma_2$.

**Definition 5 (strategies and outcomes).** *A* strategy *for* $i \in \Sigma_e$ *is a function of type* $\{r_i : r \in R^{fin}(IS)\} \to Act_i$. *We write* $S(\Gamma)$ *for the set of the vectors of strategies with one strategy for every member of* $\Gamma$ *in them. We apply the notation introduced for vectors of actions in Definitions 3 and 4 to vectors of strategies as well. Given* $s \in S(\Gamma)$ *and* $r \in R^{fin}(IS)$, *we write* $\mathrm{out}(r, s)$ *for the set*

$$\{r' = l^0 a^0 \ldots \in R^\omega(IS) : r'[0..|r|] = r, a^j_i = s_i(r_{\{i\}}[0..j]) \text{ for all } i \in \Gamma, j \geq |r|\}$$

*of the possible* outcomes *of* $r$ *when* $\Gamma$ *follow* $s$ *from time* $|r|$ *on. Given an* $X \subset R^{fin}(IS)$*, we write* $\mathrm{out}(X, s)$ *for* $\bigcup_{r \in X} \mathrm{out}(r, s)$.

**Definition 6 (indiscernibility of strategy vector sequences).** *Given* $s'$, $s'' \in S(\Sigma_e)$*, we write* $s' \sim_\Gamma s''$ *if* $s'_\Gamma = s''_\Gamma$*. Given two sequences* $s' = s'^0 \dots s'^n$, $s'' = s''^0 \dots s''^n \in (S(\Sigma_e))^{n+1}$*, we write* $s' \sim_\Gamma s''$ *if* $s'^k \sim_\Gamma s''^k$ *for* $k = 0, \dots, n$.

**Definition 7 (strategy revision).** *Given* $\Gamma \subseteq \Sigma_e$, $s', s'' \in S(\Gamma)$*, and an* $n < \omega$*, we write* $s' \triangle^n s''$ *for the vector of strategies which is defined by the case distinction:*

$$(s' \triangle^n s'')_i(r) = \begin{cases} s'_i(r_{\{i\}}), & \text{if } |r| < n; \\ s''_i(r_{\{i\}}), & \text{if } |r| \geq n. \end{cases}$$

**Definition 8 (consistency of strategy vector sequences).** *A sequence* $s = s^0, \dots, s^n \in S(\Sigma_e)^{n+1}$ *is* consistent*, if* $s^k(r) = s^{k+1}(r)$ *for all* $r \in R^{<k}(IS)$ *and all* $k < n$.

In words, $s$ is consistent, if, for $k > 0$, $s^{k+1}$ returns the same vectors of actions as $s^k$ for runs of length up to $k-1$. The reason to require $s^{k-1}(r) = s^k(r)$ only if $|r| \leq k-1$ is that, according to the definition of $\models$ in $ATL_{iR}^{DPC}$ below, for $|r| \geq k$, the values of $s^k(r)$ represent the context strategies to be followed from step $k$ on and these strategies are subject to revision.

## 2 Epistemic *ATL* with Perfect Recall, Past and Strategy Contexts ($ATL_{iR}^{DPC}$)

$ATL_{iR}^{DPC}$ has an additional parameter $\Delta$ to its game-theoretic operator to designate the set of the players whose behaviour is assumed to be as described in the strategy context. As it becomes clear below, having a cooperation modality with such a parameter facilitates the use of appropriate variants of the axioms and rules for $ATL_{iR}^{DP}$ from [9]. In Section 3 we explain that this form of the cooperation modality has the same expressive power as (an appropriately defined incomplete-information variant of) the cooperation modalities from [15].

**Definition 9 (syntax).** *Here follows a BNF for the syntax of formulas in* $ATL_{iR}^{DPC}$ *and the intended informal reading of the connectives:*

| | |
|---|---|
| $\varphi, \psi ::= \bot \mid p \mid (\varphi \Rightarrow \psi)$ | \| *logical falsehood, atomic proposition, implication* |
| $\ominus\varphi$ | \| $\varphi$ *one step ago* |
| $(\varphi \mathsf{S} \psi)$ | \| $\psi$ *either now, or some time ago and* $\varphi$ *has been true ever since* $\psi$ *held last;* |
| $\mathsf{D}_\Gamma \varphi$ | \| $\Gamma$ *know* $\varphi$; |
| $\langle\langle \Gamma \mid \Delta \rangle\rangle \circ \varphi$ | \| $\Gamma$ *can enforce* $\varphi$ *in one step, provided that* $\Delta$ *follow their current strategies;* |
| $\langle\langle \Gamma \mid \Delta \rangle\rangle (\varphi \mathsf{U} \psi)$ | \| $\Gamma$ *can enforce reaching a* $\psi$*-state along a path of* $\varphi$*-states, provided that* $\Delta$ *follow their current strategies*; |
| $[\![ \Gamma \mid \Delta ]\!] (\varphi \mathsf{U} \psi)$ | \| $\Gamma$ *cannot prevent reaching a* $\psi$*-state along a path of* $\varphi$*-states, unless* $\Delta$ *give up their current strategies.* |

*$\langle\langle\Gamma \mid \Delta\rangle\rangle$ and $[\![\Gamma \mid \Delta]\!]$ are well-formed only if $\Gamma \cap \Delta = \emptyset$. We write* $\mathrm{Var}(\varphi)$ *for the set of the atomic propositions which occur in $\varphi$.*

Note that we do not introduce dedicated notation for *individual* knowledge. Below it becomes clear that $\mathsf{K}_i$ can be written as $\mathsf{D}_{\{i\}}$.

**Definition 10 (modelling relation of $ATL_{iR}^{DPC}$).** *The relation $IS, s, r \models \varphi$ is defined for $r \in R^{fin}(IS)$, a consistent strategy vector sequence $s = s^0, \ldots, s^{|r|} \in S(\Sigma_e)^{|r|+1}$, and formulas $\varphi$, by the clauses:*

$IS, s, r \not\models \bot$;

$IS, s, l^0 a^0 \ldots a^{n-1} l^n \models p$ *iff* $V(l^n, p)$ *for atomic propositions* $p$;

$IS, s, r \models \varphi \Rightarrow \psi$ *iff either* $IS, s, r \not\models \varphi$ *or* $IS, s, r \models \psi$;

$IS, s, r \models \mathsf{D}_\Gamma \varphi$ *iff* $(\forall r' \in [r]_\Gamma)(s' \in S(\Sigma_e))(s' \sim_\Gamma s$ *implies* $IS, s', r' \models \varphi)$;

$IS, s, r \models \langle\langle\Gamma \mid \Delta\rangle\rangle\theta$ *iff*

$$(\exists s' \in S(\Gamma))(\forall s'' \in S(\Sigma_e \setminus (\Gamma \cup \Delta)))(\forall s''' \in S(\Sigma_e)^{|r|})(\forall r' \in \mathrm{out}([r]_{\Gamma\cup\Delta}, s' \cup s_\Delta^{|r|}))$$
$$(s''' \sim_{\Gamma\cup\Delta} s \text{ implies } IS, s''' \cdot (s'''^{|r|} \triangle^{|r|} (s' \cup s_\Delta^{|r|} \cup s'')), r', |r| \models \theta);$$

$IS, s, r \models [\![\Gamma \mid \Delta]\!]\theta$ *iff* $IS, s, r \not\models \langle\langle\Gamma \mid \Delta\rangle\rangle\neg\theta$;

$IS, s, r \models \ominus\varphi$ *iff* $|r| > 0$ *and* $IS, s[0..|r|-1], r[0..|r|-1] \models \varphi$;

$$IS, s, r \models (\varphi \mathsf{S} \psi) \quad \text{iff } (\exists k \leq |r|)\begin{pmatrix} IS, s[0..n-k], r[0..n-k] \models \psi \text{ and} \\ (\forall u < k) IS, s[0..n-u], r[0..n-u] \models \varphi \end{pmatrix}.$$

*In the clauses for $\langle\langle\Gamma \mid \Delta\rangle\rangle\theta$ and $[\![\Gamma \mid \Delta]\!]\theta$ above, $\theta$ stands for a possibly negated $\circ\varphi$ or $(\varphi \mathsf{U} \psi)$. We use an auxiliary form of $\models$ to define the satisfaction of $\theta$, which, being an LTL formula, takes an infinite run and a position in it to interpret. Given an $r \in R^\omega(IS)$ and a $k < \omega$,*

$$IS, s, r, k \models \circ\varphi \quad \text{iff } IS, s, r[0..k+1] \models \varphi;$$
$$IS, s, r, k \models (\varphi \mathsf{U} \psi) \text{ iff } (\exists m)\begin{pmatrix} IS, s, r[0..k+m] \models \psi \text{ and} \\ (\forall n < m) IS, s, r[0..k+n] \models \varphi \end{pmatrix};$$
$$IS, s, r, k \models \neg\theta \quad \text{iff } IS, s, r, k \not\models \theta.$$

*Validity of formulas in an entire interpreted system and on the class of all interpreted systems, that is, in the logic $ATL_{iR}^{DPC}$, is defined as satisfaction at all 0-length runs in the considered interpreted system, and at all the 0-length runs in all interpreted systems, respectively.*

In the definition of $\models$, we use *sequences* of strategy vectors and not simply strategy vectors as the strategy context, in order to enable the interpretation of the past operators $\ominus$ and $(.\mathsf{S}.)$. This complication of the form of $\models$ is inevitable because the interpretation of $\langle\langle\Gamma \mid \Delta\rangle\rangle\circ$ allows the strategy context to be revised, and it is necessary to be able to revert to contexts from before such revisions for the correct interpretation of the past operators. The semantics of the future subset of $ATL_{iR}^{DPC}$ can be defined with $s$ being just a vector of strategies in

$\models$. Then the satisfaction condition for $IS, s, r \models \langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \theta$ can be given the following simpler form:

$$(\exists s' \in S(\Gamma))(\forall s'' \in S(\Sigma_e \setminus (\Gamma \cup \Delta)))(\forall s''' \in S(\Sigma_e))(\forall r' \in \mathrm{out}([r]_{\Gamma\cup\Delta}, s' \cup s_\Delta)) \\ (s''' \sim_{\Gamma\cup\Delta} s \text{ implies } IS, s''' \triangle^{|r|}(s' \cup s_\Delta \cup s''), r', |r| \models \theta)$$

**Abbreviations.** $\top$, $\neg$, $\vee$, $\wedge$ and $\Leftrightarrow$ are used to abbreviate formulas written with $\perp$ and $\Rightarrow$ in the common way. The abbreviations below are specific to *ATL* and other temporal and epistemic logics:

$$\mathsf{I} \rightleftharpoons \neg\ominus\top \qquad \boxminus\varphi \rightleftharpoons \neg\diamondminus\neg\varphi \qquad [\![\Gamma \mid \Delta]\!] \circ \varphi \rightleftharpoons \neg\langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \neg\varphi.$$
$$\diamondminus\varphi \rightleftharpoons (\top \mathsf{S} \varphi) \qquad \mathsf{P}_\Gamma\varphi \rightleftharpoons \neg\mathsf{D}_\Gamma\neg\varphi$$

## 3 Related Work

Next we give a brief account of the systems $ATL_{sc}$, *BSIL* and *ATLES* from [15,21,20], respectively. An extension of $ATL^*$ by strategy contexts can be found in [1] too, where the authors focus mainly on modelling issues, and not technical results. We show that $ATL_{sc}$ from [15] admits a satisfaction preserving translation into $ATL_{iR}^{DPC}$. $ATL_{sc}$, *BSIL* and *ATLES* were originally introduced for alternating transition systems and concurrent game structures. To assert the semantical compatibility with $ATL_{iR}^{DPC}$ and, for the sake of brevity, we spell out their semantics on interpreted systems. Unlike $ATL_{iR}^{DPC}$, all these systems have complete information semantics. However, complete information can be straightforwardly modelled in interpreted systems by assigning the same local state space $L_1 = \ldots = L_N = L_e$ to each $i \in \Sigma_e$ and restricting the reachable states to be in the diagonal of $L_1 \times \ldots \times L_N \times L_e$.

***ATL* with strategy contexts ($ATL_{sc}$, [15])** has *state formulas* $\varphi$ and *path formulas* $\psi$. Their syntax can be given by the BNFs

$$\varphi ::= \perp \mid p \mid (\varphi \Rightarrow \varphi) \mid \langle\cdot\Gamma\cdot\rangle\psi \mid \rangle\cdot\Gamma\cdot\langle\varphi \qquad \text{and} \qquad \psi ::= \neg\psi \mid \circ\varphi \mid (\varphi \mathsf{U} \varphi)$$

Satisfaction has the form $IS, \rho, r \models \varphi$ with $r \in R^{fin}(IS)$ for state formulas and $IS, \rho, r \models \psi$ with $r \in R^\omega(IS)$ for path formulas. In both cases $\Delta \subseteq \Sigma$ and $\rho \in S(\Delta)$. The clauses about $\models$ for the $ATL_{sc}$-specific operators are as follows:

$$IS, \rho, r \models \langle\cdot\Gamma\cdot\rangle\psi \text{ iff } (\exists s \in S(\Gamma))(\forall r' \in \mathrm{out}(r, \rho_{\mathrm{dom}\rho\setminus\Gamma} \cup s)) IS, \rho_{\mathrm{dom}\rho\setminus\Gamma} \cup s, r' \models \varphi;$$
$$IS, \rho, r \models \rangle\cdot\Gamma\cdot\langle\varphi \text{ iff } IS, \rho_{\mathrm{dom}\rho\setminus\Gamma}, r \models \varphi.$$

Thanks to the presence of strategy contexts and the possibility to combine $\langle\cdot.\cdot\rangle$ with $\neg$ in $ATL_{sc}$, $ATL_{sc}$ and $ATL_{sc}^*$, where path formulas can have arbitrary combinations of boolean connectives, have the same expressive power.

The translation $t_\Delta$ below, maps from $ATL_{sc}$ state formulas to $ATL_{iR}^{DPC}$ formulas. The auxiliary parameter $\Delta \subseteq \Sigma$ is the domain of the reference context.

For the sake of brevity $(\cdot.\cdot)$ stands for either $[\cdot.\cdot]$ or $\langle\cdot.\cdot\rangle$ in the translation clauses. The meaning of $(\!(.)\!)$ is similar, wrt $\langle\!\langle .|. \rangle\!\rangle$.

$$
\begin{array}{l}
t_\Delta(\bot) \rightleftharpoons \bot, \qquad t_\Delta(p) \rightleftharpoons p, \qquad t_\Delta(\varphi_1 \Rightarrow \varphi_2) \rightleftharpoons t_\Delta(\varphi_1) \Rightarrow t_\Delta(\varphi_2) \\
t_\Delta(\langle\cdot\Gamma\cdot\rangle\neg\psi) \rightleftharpoons \neg t_\Delta([\cdot\Gamma\cdot]\psi), \qquad t_\Delta([\cdot\Gamma\cdot]\neg\psi) \rightleftharpoons \neg t_\Delta(\langle\cdot\Gamma\cdot\rangle\psi) \\
t_\Delta((\cdot\Gamma\cdot) \circ \varphi) \rightleftharpoons (\!(\Gamma \mid \Delta \setminus \Gamma)\!) \circ t_{\Gamma\cup\Delta}(\varphi) \\
t_\Delta((\cdot\Gamma\cdot)(\varphi_1 \mathsf{U} \varphi_2)) \rightleftharpoons (\!(\Gamma \mid \Delta \setminus \Gamma)\!)(t_{\Gamma\cup\Delta}(\varphi_1) \mathsf{U} t_{\Gamma\cup\Delta}(\varphi_2)) \\
t_\Delta(\cdot\rangle\Gamma\langle\cdot\varphi \mid \Delta) \rightleftharpoons t_{\Delta\setminus\Gamma}(\varphi)
\end{array}
$$

An induction on the construction of formulas shows that, for any $ATL_{sc}$ state formula $\varphi$, $IS, \rho, r \models_{ATL_{sc}} \varphi$ is equivalent to $IS, s, r \models_{ATL_{iR}^{DPC}} t_{\mathrm{dom}\rho}(\varphi)$ where $s$ stands for any sequence of strategy vectors that is consistent with $r$ and features $\rho$ as the strategy assignment to the members of dom$\rho$ in its last member. The translation can be inverted on the future subset of $ATL_{iR}^{DPC}$:

$$
t^{-1}(\langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \varphi) \rightleftharpoons \cdot\rangle \Sigma \setminus \Delta \langle\cdot\ \langle\cdot\Gamma\cdot\rangle t^{-1}(\varphi).
$$

**Basic strategy-interaction logic (*BSIL*, [21])** language includes *state* formulas $\varphi$, *path* formulas $\psi$ and *tree* formulas $\theta$:

$$
\begin{array}{l}
\varphi ::= \bot \mid p \mid (\varphi \Rightarrow \varphi) \mid \langle\Gamma\rangle\theta \mid \langle\Gamma\rangle\psi \\
\theta ::= \bot \mid (\theta \Rightarrow \theta) \mid \langle +\Gamma\rangle\theta \mid \langle +\Gamma\rangle\psi \\
\psi ::= \circ\varphi \mid (\varphi \mathsf{U} \varphi) \mid (\varphi \mathsf{W} \varphi)
\end{array}
$$

The modelling relation has the form $IS, \rho, l \models \varphi$ for state and tree $\varphi$, and $IS, \rho, r \models \psi$ for path $\psi$, where $l$ is a (global) state, $r \in R^\omega(IS)$, and $\rho$ is a strategic context. The clauses for $\langle\Gamma\rangle$ and $\langle +\Gamma\rangle$ with a path argument formula $\psi$ are as follows:

$$
\begin{array}{lll}
IS, \rho, l \models \langle\Gamma\rangle\psi & \text{iff} & (\exists s \in S(\Gamma))(\forall r \in \mathrm{out}(l, s)) IS, s, r \models \psi; \\
IS, \rho, l \models \langle +\Gamma\rangle\psi & \text{iff} & (\exists s \in S(\Gamma))(\forall r \in \mathrm{out}(l, \rho_{\mathrm{dom}\rho\setminus\Gamma} \cup s)) IS, \rho_{\mathrm{dom}\rho\setminus\Gamma} \cup s, r \models \psi
\end{array}
$$

The clauses for tree argument formulas are similar. *BSIL* admits a translation into an appropriate $^*$-extension of $ATL_{iR}^{DPC}$.

***ATL* with explicit strategies (*ATLES*, [20])** extends the syntax of $\langle\!\langle . \rangle\!\rangle$ by subscripting it with mappings $\rho$ of subsets of $\Sigma$ to finite syntactical descriptions of strategies called *strategy terms*. In our notation, $\rho$ denote elements of $S(\mathrm{dom}\rho)$ and the clause about $\models$ for $\langle\!\langle \Gamma \rangle\!\rangle_\rho \circ$ is:

$$
IS, r \models \langle\!\langle \Gamma \rangle\!\rangle_\rho \circ \varphi \text{ iff } (\exists s \in S(\Gamma \setminus \mathrm{dom}\rho))(\forall r' \in \mathrm{out}(r, s \cup \rho)) IS, r'[0..|r|+1] \models \varphi.
$$

The clauses for $\langle\!\langle \Gamma \rangle\!\rangle_\rho(\varphi \mathsf{U} \psi)$ and $\langle\!\langle \Gamma \rangle\!\rangle_\rho \Box\varphi$ follow the same pattern. Unlike $ATL_{sc}$, *BSIL* and $ATL_{iR}^{DPC}$, an *ATLES* formula may have several (freely occurring) fixed strategy context terms for each player. There appears to be no obvious way to reconcile this with the semantics of the other systems.

## 4 A Proof System for Basic $ATL_{iR}^{DPC}$

*Basic* $ATL_{iR}^{DPC}$ is the subset of $ATL_{iR}^{DPC}$ without $\langle\!\langle . \mid . \rangle\!\rangle (.\mathsf{U}.)$ and $[\![ . \mid . ]\!] (.\mathsf{U}.)$:

$$\varphi, \psi ::= \bot \mid p \mid (\varphi \Rightarrow \psi) \mid \ominus\varphi \mid (\varphi \mathsf{S} \psi) \mid \mathsf{D}_\Gamma \varphi \mid \langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \varphi$$

Along with all propositional tautologies and the rule *Modus Ponens* ($MP$), our system includes the following axioms and rules:

*The epistemic operator* $\mathsf{D}_.$

($\mathbf{K}_\mathsf{D}$) $\mathsf{D}_\Gamma(\varphi \Rightarrow \psi) \Rightarrow (\mathsf{D}_\Gamma \varphi \Rightarrow \mathsf{D}_\Gamma \psi)$ ($\mathbf{T}_\mathsf{D}$) $\mathsf{D}_\Gamma \psi \Rightarrow \psi$

($\mathbf{4}_\mathsf{D}$) $\mathsf{D}_\Gamma \psi \Rightarrow \mathsf{D}_\Gamma \mathsf{D}_\Gamma \psi$ ($\mathbf{5}_\mathsf{D}$) $\neg \mathsf{D}_\Gamma \psi \Rightarrow \mathsf{D}_\Gamma \neg \mathsf{D}_\Gamma \psi$

($Mono_\mathsf{D}$) $\mathsf{D}_\Gamma \psi \Rightarrow \mathsf{D}_{\Gamma \cup \Delta} \psi$ ($N_\mathsf{D}$) $\dfrac{\varphi}{\mathsf{D}_\Gamma \varphi}$

($INT_\mathsf{D}$) $\dfrac{(\mathsf{D}_{\Gamma \setminus \Delta}(p \Rightarrow \varphi) \wedge \mathsf{D}_\Delta(\neg p \Rightarrow \varphi)) \Rightarrow \psi}{\mathsf{D}_{\Gamma \cup \Delta} \varphi \Rightarrow \psi}$

*The past modalities* $\ominus$ *and* $(.\mathsf{S}.)$

($\mathbf{K}_\ominus$) $\ominus(\varphi \Rightarrow \psi) \Rightarrow (\ominus\varphi \Rightarrow \ominus\psi)$ ($\ominus\bot$) $\neg\ominus\bot$

($FP_{(.\mathsf{S}.)}$) $(\varphi \mathsf{S} \psi) \Leftrightarrow \psi \vee (\varphi \wedge \ominus(\varphi \mathsf{S} \psi))$ ($Fun_\ominus$) $\ominus\neg\varphi \Rightarrow \neg\ominus\varphi$

($Mono_\ominus$) $\dfrac{\varphi \Rightarrow \psi}{\ominus\varphi \Rightarrow \ominus\psi}$ ($N_\boxminus$) $\dfrac{\varphi}{\boxminus\varphi}$

*General ATL axioms and rules*

($\langle\!\langle . \mid . \rangle\!\rangle \circ \bot$) $\neg \langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \bot$

($\langle\!\langle . \mid . \rangle\!\rangle \circ \top$) $\langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \top$

($S$) $\langle\!\langle \Gamma' \setminus \Gamma'' \mid \Delta' \rangle\!\rangle \circ \varphi \wedge \langle\!\langle \Gamma'' \mid \Delta'' \rangle\!\rangle \circ \psi \Rightarrow \langle\!\langle \Gamma' \cup \Gamma'' \mid \Delta' \cup \Delta'' \rangle\!\rangle \circ (\varphi \wedge \psi)$

($INT_{\langle\!\langle . \mid . \rangle\!\rangle \circ}$) $\dfrac{\langle\!\langle \Gamma' \setminus \Gamma'' \mid \Delta' \rangle\!\rangle \circ (p \Rightarrow \varphi) \wedge \langle\!\langle \Gamma'' \mid \Delta'' \rangle\!\rangle \circ (\neg p \Rightarrow \varphi) \Rightarrow \psi}{\langle\!\langle \Gamma' \cup \Gamma'' \mid \Delta' \cup \Delta'' \rangle\!\rangle \circ \varphi \Rightarrow \psi}$

($Mono_{\langle\!\langle . \mid . \rangle\!\rangle \circ}$) $\dfrac{\varphi \Rightarrow \psi}{\langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \varphi \Rightarrow \langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \psi}$

*Committed versus neutral players* (See Lemma 1 from [20].)

($WHW$) $\langle\!\langle \Gamma \mid \Psi \cup \Delta \rangle\!\rangle \circ \varphi \Rightarrow \langle\!\langle \Gamma \cup \Psi \mid \Delta \rangle\!\rangle \circ \varphi$

*Interactions between* $\ominus$, $(.\mathsf{S}.)$, $\langle\!\langle . \mid . \rangle\!\rangle \circ$ *and* $\mathsf{D}_.$

($\mathsf{D}_\circ$) $\langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \varphi \Leftrightarrow \mathsf{D}_{\Gamma \cup \Delta} \langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \varphi$

$\langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \varphi \Leftrightarrow \langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \mathsf{D}_{\Gamma \cup \Delta} \varphi$

($PR$) $\ominus \mathsf{D}_\Gamma \varphi \Rightarrow \mathsf{D}_\Gamma \ominus \varphi$

($\langle\!\langle . \mid . \rangle\!\rangle \circ \ominus$) $\langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ (\ominus\varphi \wedge \psi) \Leftrightarrow \mathsf{D}_{\Gamma \cup \Delta} \varphi \wedge \langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \psi$

$\ominus \langle\!\langle \emptyset \mid \Delta \rangle\!\rangle \circ \varphi \Rightarrow \langle\!\langle \emptyset \mid \Delta \rangle\!\rangle \circ \ominus\varphi$

($\mathsf{D}_\mathsf{I}$) $\mathsf{D}_\Gamma \mathsf{I} \vee \mathsf{D}_\Gamma \neg \mathsf{I}$

The rules $INT_{\langle\langle .|.\rangle\rangle\circ}$ and $INT_{\mathsf{D}}$ require $p \notin \mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi)$. Note that instances of $S$ are well-formed only if $(\Gamma' \cup \Gamma'') \cap (\Delta' \cup \Delta'') = \emptyset$.

## 5 Completeness of the Proof System

We fix the vocabulary $AP$ for the rest of this section and denote the set of all the basic $ATL_{iR}^{DPC}$ formulas built using variables from $AP$ by $\mathbf{L}$. We write $\Phi \vdash_{MP} \varphi$ for the derivability of $\varphi$ from the premises $\Phi$, the theorems of $ATL_{iR}^{DPC}$ and $MP$ as the only proof rule.

**Auxiliary Propositional Variables and Formulas.** Given $\Gamma \subseteq \Sigma$ and $i \in \Sigma$, we write $\Gamma^{<i}$ for the set $\Gamma \cap \{1, \ldots, i-1\}$. Given the formulas $\mathsf{D}_\Gamma\psi$ and $\langle\langle \Gamma \mid \Delta\rangle\rangle \circ \psi$, we introduce the auxiliary variables $q_{i,\mathsf{D}_\Gamma\psi}$, $i \in \Gamma^{<\max\Gamma}$, and $q_{i,\langle\langle \Gamma|\Delta\rangle\rangle\circ\psi}$, $i \in (\Gamma \cup \Delta)^{<\max(\Gamma\cup\Delta)}$ and use them to construct the formulas

$$p_{i,\mathsf{D}_\Gamma\psi} \rightleftharpoons q_{i,\mathsf{D}_\Gamma\psi} \wedge \bigwedge_{j\in\Gamma^{<i}} \neg q_{j,\mathsf{D}_\Gamma\psi},\ i \in \Gamma^{<\max\Gamma},\ \text{and } p_{\max\Gamma,\mathsf{D}_\Gamma\psi} \rightleftharpoons \bigwedge_{j\in\Gamma^{<\max\Gamma}} \neg q_{j,\mathsf{D}_\Gamma\psi}.$$

Obviously these formulas satisfy $\vdash \bigvee_{i\in\Gamma} p_{i,\mathsf{D}_\Gamma\psi}$ and $\vdash \neg(p_{i,\mathsf{D}_\Gamma\psi} \wedge p_{j,\mathsf{D}_\Gamma\psi})$ for $i \neq j$. We put $p_{\max\Gamma,\mathsf{D}_\Gamma\psi} \rightleftharpoons \top$ in case $|\Gamma \cup \Delta| = 1$. We use $p_{i,\mathsf{D}_\Gamma\psi}$ to construct the formulas

$$\mathsf{D}_{i,\Gamma}\psi \rightleftharpoons \mathsf{D}_i(p_{i,\mathsf{D}_\Gamma\psi} \Rightarrow \psi),\ i \in \Gamma.$$

The formulas $p_{i,\langle\langle\Gamma|\Delta\rangle\rangle\circ\psi}$, $i \in \Gamma\cup\Delta$, are written similarly in terms of the variables $q_{i,\langle\langle\Gamma|\Delta\rangle\rangle\circ\psi}$, $i \in (\Gamma \cup \Delta)^{<\max(\Gamma\cup\Delta)}$. We use $p_{i,\langle\langle\Gamma|\Delta\rangle\rangle\circ\psi}$ to construct the formulas

$$\langle\langle i, \Gamma \mid \Delta\rangle\rangle \circ \psi \rightleftharpoons \begin{cases} \langle\langle i \mid \emptyset\rangle\rangle \circ (p_{i,\langle\langle\Gamma|\Delta\rangle\rangle\circ\psi} \Rightarrow \psi), \text{ for } i \in \Gamma; \\ \langle\langle \emptyset \mid i\rangle\rangle \circ (p_{i,\langle\langle\Gamma|\Delta\rangle\rangle\circ\psi} \Rightarrow \psi), \text{ for } i \in \Delta. \end{cases}$$

Given a set of formulas $x$ written in $AP$, we write $\overline{x}$ for the set

$$x \cup \{\mathsf{D}_{i,\Gamma}\psi : i \in \Gamma, \mathsf{D}_\Gamma\psi \in x\} \cup \{\langle\langle i, \Gamma \mid \Delta\rangle\rangle \circ \psi : \langle\langle \Gamma \mid \Delta\rangle\rangle \circ \psi \in x, i \in \Gamma \cup \Delta\}.$$

**Lemma 1.** *Let $x$ be a consistent set of formulas written in $AP$. Then $\overline{x}$ is consistent too.*

The proof of this lemma is similar to that of Lemma 12 from [9] and involves the rules $INT_{\mathsf{D}}$ and $INT_{\langle\langle .|.\rangle\rangle\circ}$.

**Lemma 2 (customized Lindenbaum lemma).** *Let $\prec$ be a well-ordering of $\mathbf{L}$ and let $x$ be a consistent subset of $\mathbf{L}$. Then there exists a consistent set $x' \supseteq x$ which is maximal in $\mathbf{L}$ and is such that for any initial interval $\Phi \subset \mathbf{L}$ of $\langle \mathbf{L}, \prec\rangle$ the consistency of $x \cup \Phi$ entails $\Phi \subseteq x'$.*

*Proof.* We construct the ascending (transfinite) sequence $x_\varphi$, $\varphi \in \mathbf{L}$, of consistent subsets of $\mathbf{L}$ indexed by the elements of $\mathbf{L}$ by induction on the well-ordering $\prec$. Let $\varphi_0$ be the least element of $\mathbf{L}$. Then $x_{\varphi_0}$ is $x \cup \{\varphi_0\}$ in case $x \cup \{\varphi_0\}$ is consistent; otherwise it is just $x$. Similarly, for all non-limit $\varphi$, given that $\varphi'$ is the successor of $\varphi$ in $\prec$, $x_{\varphi'}$ is $x_\varphi \cup \{\varphi'\}$ in case $x_\varphi \cup \{\varphi'\}$ is consistent and $x_\varphi$ otherwise. For limit $\varphi$, $x_\varphi = \bigcup_{\psi\prec\varphi} \varphi_\psi$. A direct check shows that $x' = \bigcup_{\varphi\in\mathbf{L}}$ has the desired property.

In the sequel, given $x$ and a well-ordering $a$ of $\mathbf{L}$, we denote a fixed maximal consistent set (MCS) with the above property by $x + a$.

Next we build a interpreted system $IS = \langle\langle L_i : i \in \Sigma_e\rangle, I, \langle Act_i \in \Sigma_e\rangle, t, V\rangle$ for basic $ATL_{iR}^{DPC}$ which is canonical in the sense adopted in modal logic.

**Definition 11 (global states, local states).** *Let $W$ to be the set of all the maximal consistent sets of formulas in the vocabulary $AP$. Given $w \in W$ and $\Gamma \subseteq \Sigma$, we put*

$$D_\Gamma(w) = \{\varphi : \mathsf{D}_\Gamma\varphi \in \overline{w}\},\ L_i = \{D_{\{i\}}(w) : w \in W\} \text{ for } i \in \Sigma, \text{ and } L_e = W.$$

Given $w \in W$, we write $l_w$ for the state $\langle D_{\{1\}}(w), \ldots, D_{\{N\}}(w), w\rangle$. Below it becomes clear that all reachable states in $IS$ have this form. We work with the MCS $w$ instead of the respective tuples $l_w$ wherever this is more convenient. Note that the environment component of $l_w$ is $w$ itself.

**Definition 12 (valuation and initial states).** *We put $V(w, p) \leftrightarrow p \in w$ and $I = \{l_w : w \in W, \mathsf{I} \in w\}$.*

**Definition 13 (indiscernibility of states in terms of MCS).** *Given $\Gamma \subseteq \Sigma_e$, two states $w, v \in W$ are* $\Gamma$-similar*, written $w \sim_\Gamma v$, if $D_\Gamma(w) = D_\Gamma(v)$.*

The following lemma shows that $w \sim_\Gamma v$ is equivalent to $l_w \sim_\Gamma l_v$ in the sense of Definition 3.

**Lemma 3 ($\Gamma$-similarity in terms of $\Gamma$'s distributed knowledge).** *Let $w, v \in W$, $\Gamma \subseteq \Sigma$. Then $w \sim_\Gamma v$ iff $D_{\{i\}}(w) = D_{\{i\}}(v)$ for all $i \in \Gamma$.*

*Proof.* ($\leftarrow$): Let $w \sim_\Gamma v$ and $\mathsf{D}_\Gamma\varphi \in D_\Gamma(w)$. Then $\mathsf{D}_{i,\Gamma}\varphi \in \overline{w}$ for every $i \in \Gamma$. Then, by $\mathbf{4}_\mathsf{D}$ and $Mono_\mathsf{D}$, $\mathsf{D}_\Gamma\mathsf{D}_{i,\Gamma}\varphi \in \overline{v}$ for every $i \in \Gamma$ too. Now **S5** reasoning and $\vdash \bigvee\limits_{i\in\Gamma} p_{i,\mathsf{D}_\Gamma\varphi}$ entail $\bigwedge\limits_{i\in\Gamma} \mathsf{D}_{i,\Gamma} \Rightarrow \mathsf{D}_\Gamma\varphi$. Hence $\mathsf{D}_\Gamma\varphi \in D_\Gamma(v)$.
($\rightarrow$): Let $\mathsf{D}_i\varphi \in \overline{w}$. Then, by **S5** reasoning, $\overline{w} \vdash_{MP} \mathsf{D}_\Gamma\mathsf{D}_i\varphi$, whence, by $D_\Gamma(w) = D_\Gamma(v)$, $\overline{v} \vdash_{MP} \mathsf{D}_\Gamma\mathsf{D}_i\varphi$ and, finally $\mathsf{D}_i\varphi \in \overline{v}$. Hence $D_{\{i\}}(w) \subseteq D_{\{i\}}(v)$.

The symmetrical inclusions are proved similarly.

**Definition 14 (actions).** *An* action for player $i \in \Sigma$ *is either the symbol* $\mathsf{d}$ *or a tuple of the form $\langle\Phi, \Gamma, \Delta\rangle$ such that $\Gamma, \Delta \subseteq \Sigma$ are disjoint, $i \in \Gamma$, and $\Phi$ is a consistent set of formulas. An* environment action *is a well-ordering of* $\mathbf{L}$.

Player action $\langle\Phi, \Gamma, \Delta\rangle$ is represents the player's contribution to achieving all the objectives from $\Phi$ simultaneously as a member of $\Gamma$, provided that $\Delta$ act as described in the context. Action $\mathsf{d}$ indicates choosing to follow the strategy from the context. Allowing infinite sets $\Phi$ of objectives in actions is necessary because MCS may contain infinitely many formulas of the form $\langle\!\langle\emptyset \mid \Delta\rangle\!\rangle \circ \varphi$.

**Definition 15 (the past of a state).** *Given a $w \in W$, we write $\Theta_w$ for the set $\{\ominus\theta : \theta \in w\}$.*

The formulas from $\Theta_w$ hold at states which can be reached from $l_w$ in one step. Environment actions complement the construction of successor states. Let $x$ consist of the formulas to be satisfied due to the player actions performed at state $l_w$. Then environment action $a_e$ comlements $x \cup \Theta_w$ to an MCS description of the successor state $(x \cup \Theta_w) + a_e$. Lemma 2 entails that any MCS $x' \supseteq x \cup \Theta_w$ has the form $(x \cup \Theta_w) + a_e$ for some appropriate $a_e$.

**Definition 16 (effectiveness of coalitions).** *Let $a \in Act_{\Sigma_e}$ and $w \in W$. We write $\Delta_a$ for the set $\{i \in \Sigma : a_i = \mathsf{d}\}$. Coalition $\Gamma \subseteq \Sigma$ is* effective in $a$ wrt $w$, *if*

*(1) $\Gamma \subseteq \Sigma \setminus \Delta_a$, $\Delta \subseteq \Delta_a$;*

*assuming that $a_i = \langle \Phi_i, \Gamma_i, \Delta_i \rangle$, $i \in \Gamma$,*

*(2) $\Gamma = \Gamma_i$ for all $i \in \Gamma$;*

*(3) $\langle\langle \Gamma_i \mid \Delta_i \rangle\rangle \circ \bigwedge \Phi' \in w$ for all finite $\Phi' \subseteq \Phi_i$;*

*(4) $\Delta_i = \Delta_j$ for all $i, j \in \Gamma$.*

Coalition $\Gamma$ is effective in state $w$, iff its objectives are achievable in $w$. Different coalitions which are effective in the same state cannot overlap.

**Definition 17 ($\overline{a}_w$).** *Let the coalitions which are effective in $a \in Act_{\Sigma_e}$ wrt $w \in W$ be $\Gamma_1, \ldots, \Gamma_k$. Let $\Upsilon = \Gamma_1 \cup \ldots \cup \Gamma_k$. We define $\overline{a}_w \in Act_{\Sigma_e}$ by the clause*

$$\overline{a}_{w,i} = \begin{cases} a_e, & \text{if } i = e; \\ \langle \Phi_i, \Gamma_i, \Delta_i \rangle, & \text{if } i \in \Upsilon; \\ \langle \{\psi : \langle\langle \emptyset \mid i \rangle\rangle \circ \psi \in \overline{w}\}, \{i\}, \emptyset \rangle, & \text{if } i \in \Delta_a; \\ \langle \emptyset, \{i\}, \emptyset \rangle, & \text{otherwise.} \end{cases}$$

The vector of actions $\overline{a}_w$ is a revision of $a$ according to the plausibility of the actions from $a$ in $w$. In $\overline{a}_w$, players $i$ who participate in effective coalitions are described as acting to achieve the common objective of their coalitions; players who follow their respective strategies from the context are described as acting to achieve whatever consequences these actions have according to $w$, and players who neither participate in effective coalitions, nor follow the context strategies, are described as acting in singleton coalitions $\{i\}$ to achieve nothing. Consequently, all the coalitions in $\overline{a}_w$ are effective wrt $w$:

**Lemma 4 (effectiveness of coalitions in $\overline{a}_w$).** *Assuming the notation from Definition 17, if $i \in \Sigma$ and $\overline{a}_{w,i} = \langle \Phi, \Gamma, \Delta \rangle$, then $\langle\langle \Gamma \mid \Delta \rangle\rangle \circ \bigwedge \Phi' \in w$ for all finite $\Phi' \subset \Phi$.*

*Proof.* The lemma follows immediately for $i \in \Upsilon$ and $i \in \Sigma \setminus (\Upsilon \cup \Delta_a)$. Players $i \in \Delta_a$ appear in the singleton coalitions $\{i\}$ in $\overline{a}_w$, and we have $\langle\langle \emptyset \mid i \rangle\rangle \circ \bigwedge \Phi' \in w$ for all finite $\Phi' \subset \Phi$. By Axiom $WHW$ this entails $\langle\langle i \mid \emptyset \rangle\rangle \circ \bigwedge \Phi' \in w$.

As it becomes clear below, $a$ and $\overline{a}_w$ cause the same transitions from $w$. The notation $\overline{a}_w$ is introduced to avoid lengthy explanations about treating the various sorts of actions separately.

**Definition 18 (transition function).** *Let $w \in W$, $a \in Act_{\Sigma_e}$ and $\overline{a}_w = \langle \langle \Phi_i, \Gamma_i, \Delta_i \rangle : i \in \Sigma \rangle \cup \langle a_e \rangle$. Then $t(l_w, a) = l_v$ where $v = \left( \bigcup\limits_{i \in \Sigma} \Phi_i \cup \Theta_w \right) + a_e$,* *i.e., $v$ is the extension of the set of all the objectives which can be simultaneously achieved by the coalitions which are effective in $a$ wrt $w$ and the formulas which describe $l_w$'s past, to an MCS by $a_e$, as in Lemma 2.*

The definition of $t$ above relies on the fact that $\bigcup_{i\in\Sigma} \Phi_i \cup \Theta_w$ is consistent. To realise that, assume the contrary. Then there exist some finite $\Phi' \subseteq \bigcup_{i\in\Sigma} \Phi_i$ and $\Theta' \subset w$ such that $\vdash \bigwedge \Phi' \Rightarrow \ominus\neg \bigwedge \Theta'$. By the monotonicity of $\langle\!\langle \Sigma \mid \emptyset \rangle\!\rangle\circ$ and Axiom $\langle\!\langle . \mid . \rangle\!\rangle \circ \ominus$, this entails $\vdash \langle\!\langle \Sigma \mid \emptyset \rangle\!\rangle \circ \bigwedge \Phi' \Rightarrow \mathsf{D}_\Sigma \neg \bigwedge \Theta'$. By Axioms $S$ and $WHW$, $\vdash \bigwedge_{i\in\Sigma} \langle\!\langle \Gamma_i \mid \Delta_i \rangle\!\rangle \circ \bigwedge(\Phi_i \cap \Phi') \Rightarrow \langle\!\langle \Sigma \mid \emptyset \rangle\!\rangle \circ \bigwedge \Phi'$. Hence $\vdash \bigwedge_{i\in\Sigma} \langle\!\langle \Gamma_i \mid \Delta_i \rangle\!\rangle \circ \bigwedge(\Phi_i \cap \Phi') \Rightarrow \mathsf{D}_\Sigma \neg \bigwedge \Theta'$, which is a contradiction because $\Theta' \subset w$ and, by Lemma 4, $\langle\!\langle \Gamma_i \mid \Delta_i \rangle\!\rangle \circ \bigwedge(\Phi_i \cap \Phi') \in w$ for all $i \in \Sigma$. We define $t$ only on states of the form $l_w$. The set of these states contains $I$ and is closed under $t$. Hence the definition of $t$ on other states is irrelevant.

Definitions 11, 12, 14 and 18 give a complete description of the interpreted system $IS$. Below we prove that if $r \in R^{fin}(IS)$ and $s \in S(\Sigma_e)^{n+1}$ is consistent with $s$, then, for any $\varphi \in \mathbf{L}$, $IS, s, r \models \varphi$ iff $\varphi \in w$ where $l_w$ is the last state of $r$.

**Definition 19 (extracting strategic context from MCS).** *Given a $w \in R^0(IS)$, we define $a_w \in Act_\Sigma$ by putting $a_{w,i} = \langle \{\varphi : \langle\!\langle \emptyset | i \rangle\!\rangle \circ \varphi \in \overline{w}\}, \{i\}, \emptyset \rangle$. We define the vector of strategies $s_{IS} \in S(\Sigma)$ by putting, given an arbitrary $r = w^0 a^0 \dots a^{|r|-1} w^{|r|} \in R^{fin}(IS)$, $s_{IS}(r) = a_{w^{|r|}}$.*

The strategies $s_{IS}$ are built according to the working of the transition function along runs in which the players act as described in the context. They are memoryless, i.e., determined by the last state of the argument run. Note that we extract strategic context from $\overline{w}$, which contains the explicit descriptions $\langle\!\langle i, \Gamma \mid \Delta \rangle\!\rangle \circ \psi$ of the contribution of individual coalition members $i \in \Gamma$ to the achievement of goals of their respective coalitions $\Gamma$. According to our definition, local runs are sequences of the form

$$r_i = D_{\{i\}}(w^0) a_i^0 D_{\{i\}}(w^1) a_i^1 \dots a_i^{k-1} D_{\{i\}}(w^k) \dots \ .$$

To realise that the strategies $s_{IS,i}$ are determined from the local run of player $i$, note that $\langle\!\langle \emptyset | \{i\} \rangle\!\rangle \circ \varphi \in \overline{w}$ is equivalent to $\langle\!\langle \emptyset | \{i\} \rangle\!\rangle \circ \varphi \in \overline{v}$ for $v$ such that $D_{\{i\}}(v) = D_{\{i\}}(w)$ due to the Axioms $\mathsf{D}_\circ$.

**Definition 20 (consistency between runs and strategy vector sequences).** *Let $n < \omega$. Run $r = w^0 a^0 \dots a^{n-1} w^n \in R^n(IS)$ and strategy vector sequence $s = s^0, \dots, s^n \in (S(\Sigma_e))^{n+1}$ are* consistent, *if $s$ is consistent (in the sense of Definition 8) and $a^k = s^n(r[0..k])$, $k = 0, \dots, n-1$.*

Note that the restrictions on $s^k$ which follow from the consistency between $r$ and $s$ for $k < n$ are implied by the consistency of $s$ as a sequence of strategy vectors.

The following lemma states that if $w \in W$ and $\mathsf{D}_\Gamma(w)$ is consistent with some arbitrary formula $\varphi$, then indeed $D_\Gamma(w) \vdash_{MP} \mathsf{P}_\Gamma \varphi$.

**Lemma 5.** *Let $\Gamma \subseteq \Sigma$, $w \in W$, $\varphi \in \mathbf{L}$ and let $w$ be consistent with $\varphi$. Then $D_\Gamma(w) \vdash_{MP} \mathsf{P}_\Gamma \varphi$.*

*Proof.* Since $w$ is maximal consistent, either $\mathsf{P}_\Gamma\varphi \in w$ or $\mathsf{D}_\Gamma\neg\varphi \in w$. The latter is impossible as it would entail $D_\Gamma(w) \vdash_{MP} \neg\varphi$. By **S5** reasoning, $\mathsf{P}_\Gamma\varphi \in w$ entails $\mathsf{D}_\Gamma\mathsf{P}_\Gamma\varphi \in w$. The latter formula appears in $D_\Gamma(w)$ as well. Hence, by **S5** reasoning again, $D_\Gamma(w) \vdash_{MP} \mathsf{P}_\Gamma\varphi$.

**Lemma 6.** *Let $w, v \in W$, $\Gamma \subseteq \Sigma$ and $D_\Gamma(w) \subseteq D_\Gamma(v)$. Then $D_\Gamma(w) = D_\Gamma(v)$.*

*Proof.* Let $\varphi \in \mathbf{L}$ be such that $\mathsf{D}_\Gamma\varphi \notin w$. Then $\mathsf{D}_\Gamma\mathsf{P}_\Gamma\neg\varphi \in w$ by Lemma 5 and **S5** reasoning. By $D_\Gamma(w) \subseteq D_\Gamma(v)$, this entails $\mathsf{P}_\Gamma\neg\varphi \in v$. Hence $\mathsf{D}_\Gamma\varphi \notin v$.

The following lemmata are the parts of the inductive proof of the Truth Lemma below (Theorem 1) about the various $ATL_{iR}^{DPC}$ modalities.

**Lemma 7** ($\mathsf{P}_\Gamma$)**.** *Let $n < \omega$, $r = w^0a^0\dots a^{n-1}w^n \in R^n(IS)$, let $s = s^0, \dots, s^n \in (S(\Sigma_e))^{n+1}$ be consistent with $r$. Let $\Phi$ be a set of formulas such that $\mathsf{P}_\Gamma \bigwedge \Phi' \in w^n$ for all finite $\Phi' \subseteq \Phi$. Then there exist an $r' = v^0b^0\dots b^{n-1}v^n \in R^{fin}(IS)$ and a sequence $s' = s'^0, \dots, s'^n \in (S(\Sigma_e))^{n+1}$ such that $s'$ is consistent with $r'$, $r' \sim_\Gamma r$, $s' \sim_\Gamma s$, and $\Phi \subseteq v^n$.*

*Proof.* Induction on $n$. If $r \in R^0(IS)$, then $\mathsf{I} \in w^0$ and, by **S5** reasoning and $\mathsf{D}_\mathsf{I}$, $\mathsf{P}_\Gamma(\mathsf{I} \wedge \bigwedge\Phi') \in w^0$ for all finite $\Phi' \subseteq \Phi$. By **S5** reasoning this entails that $\Phi \cup \{\mathsf{I}\} \cup \{\mathsf{D}_\Gamma\psi : \mathsf{D}_\Gamma\psi \in w^0\}$ is consistent. Now Lemma 6 entails that we can choose $v^0$ to be any MCS which contains the latter set and put $s'^0 = s_{IS}$.

Next we prove the lemma for $|r| = n+1$ assuming that it holds for $|r| = n$. Let $\Theta = \{\theta : D_\Gamma(w^{n+1}) \cup \Phi \vdash_{MP} \ominus\theta\}$. Assume that $D_\Gamma(w^n) \cup \Theta$ is inconsistent for the sake of contradiction. Then there exist some finite $D' \subset D_\Gamma(w^n)$ and $\Theta' \subset \Theta$ such that $\vdash \mathsf{D}_\Gamma\ominus \bigwedge D' \Rightarrow \ominus\neg\bigwedge\Theta'$. By Axiom $PR$, $\mathsf{D}_\Gamma\ominus\bigwedge D' \in D_\Gamma(w^{n+1})$. Hence $\neg\bigwedge\Theta' \in \Theta$, which entails that $D_\Gamma(w^{n+1}) \cup \Phi$ is inconsistent. This means that there exists a finite $\Phi' \subseteq \Phi$ such that $D_\Gamma(w^{n+1}) \vdash_{MP} \mathsf{D}_\Gamma\neg\bigwedge\Phi'$, which is a contradiction. Hence $D_\Gamma(w^n) \cup \Theta$ is consistent, and consequently, because of the closedness of $\Theta$ under conjunction, $\{\mathsf{P}_\Gamma\bigwedge\Theta' : \Theta' \subset_{fin} \Theta\} \subset w^n$.

By the inductive hypothesis, there exist an $r'' = v^0b^0\dots b^{n-1}v^n \in R^{fin}(IS)$ and a sequence $s'' = s''^0, \dots, s''^n \in (S(\Sigma_e))^{n+1}$ such that $s''$ is consistent with $r''$, $r'' \sim_\Gamma r[0..n]$, $s'' \sim_\Gamma s[0..n]$, and $\Theta \subseteq v^n$. Next we define $s'^n s'^{n+1} \in S(\Sigma_e)$ so that $s' = s''[0..n-1] \cdot s'^n, s'^{n+1}$ is consistent with $r' = r''\, b^n\, v^{n+1} \in R^{n+1}(IS)$ where $b^n = s'^n(r'')$, $v^{n+1} = t(v^n, b^n)$, $s' \sim_\Gamma s$, $\mathsf{D}_\Gamma(v^{n+1}) = \mathsf{D}_\Gamma(w^{n+1})$ and $\Phi \subseteq v^{n+1}$. Given $g \in R^{fin}(IS)$, we put

$$s'^n_i(g) = \begin{cases} s'^{n-1}_i(g), & \text{if } |g| < n, \text{ for all } i \in \Sigma_e, \\ s^n_i(g) & \text{if } |g| = n, \text{ and } i \in \Gamma, \\ \langle i, \emptyset, \{i\}, \emptyset \rangle & \text{if } |g| = n, \text{ and } i \in \Sigma \setminus \Gamma, \\ \text{any well-ordering of } \mathbf{L} \text{ in which} & \\ \mathsf{D}_\Gamma(w^{n+1}) \cup \Phi \text{ forms an initial interval,} & \text{if } |g| = n, \text{ and } i = e, \\ s_{IS,i}(g), & \text{if } |g| \geq n. \end{cases}$$

For $s'^{n+1}$ we put $s'^{n+1}(g) = \begin{cases} s''^n(g), \text{ if } |g| < n+1, \\ s_{IS}(g), \text{ if } |g| \geq n+1. \end{cases}$

By construction, $s'$ is consistent (as a sequence of strategy vectors), $s' \sim_\Gamma s$, and $s'$ is consistent with $r''$. We need to prove that $\mathsf{D}_\Gamma(v^{n+1}) = \mathsf{D}_\Gamma(w^{n+1})$ and $\Phi \subseteq v^{n+1}$. Note that, by **S5**-reasoning, $\mathsf{D}_\Gamma(w^{n+1}) \subset v^{n+1}n$ entails $\mathsf{D}_\Gamma(w^{n+1}) = \mathsf{D}_\Gamma(v^{n+1})$. This is so, because $w^{n+1}$ is a MCS, whence, if $\mathsf{D}_\Gamma\psi \in v^{n+1} \setminus w^{n+1}$, then, $\mathsf{D}_\Gamma\neg\mathsf{D}_\Gamma\psi \in w^{n+1}$ follows by negative introspection from $\neg\mathsf{D}_\Gamma\psi \in w^{n+1}$. The latter entails $\mathsf{D}_\Gamma\neg\mathsf{D}_\Gamma\psi \in v^{n+1}$, which, by **T**, entails $\neg\mathsf{D}_\Gamma\psi \in v^{n+1}$, and this would contradict the consistency of $v^{n+1}$. Hence we only need to prove that $\Phi \subseteq v^{n+1}$. By the definition of the transition function $t$, this would follow from the consistency of $\bigcup\limits_{i\in\Gamma} \Phi_i \cup \{\ominus\theta : \theta \in v^n\} \cup D_\Gamma(w^{n+1}) \cup \Phi$ where $\Phi_i$ are the sets of formulas occurring in $\overline{b}^n_{v^n,\Gamma}$. This boils down to the consistency of $\{\ominus\theta : \theta \in v^n\} \cup D_\Gamma(w^{n+1}) \cup \Phi$, because $v^n \sim_\Gamma w^n$ entails that $a^n$ has the same effective $\Gamma_i \subseteq \Gamma$ wrt $w$ as $b^n$ wrt $v^n$, and therefore $\Phi_i$, $i \in \Gamma$, are the same in $\overline{a}^n_{w^n,\Gamma}$. Hence, by the definition of $w^{n+1} = t(w^n, a^n)$ and $\mathsf{D}_\circ$, $\psi \in \bigcup\limits_{i\in\Gamma} \Phi_i$ is equivalent to $\mathsf{D}_\Gamma\psi \in D_\Gamma(w^{n+1})$. Now let us assume that $\{\ominus\theta : \theta \in v^n\} \cup D_\Gamma(w^{n+1}) \cup \Phi$ is inconsistent for the sake of contradiction. Then there exist some finitely many $\theta_1, \ldots, \theta_k \in v^n$ such that $D_\Gamma(w^{n+1}) \cup \Phi \vdash_{MP} \circ\neg(\theta_1 \wedge \ldots \wedge \theta_k)$. This is impossible by the choice of $v^n$ to be a superset of $\Theta = \{\theta : D_\Gamma(w^{n+1}) \cup \Phi \vdash_{MP} \ominus\theta\}$.

**Lemma 8 ($\mathsf{D}_\Gamma$).** *Let $n < \omega$, $r, r' \in R^n(IS)$. Let $r = w^0a^0 \ldots a^{n-1}w^n$ and $r' = v^0b^0 \ldots b^{n-1}v^n$. Then $r' \sim_\Gamma r$ implies $\varphi \in v^n$.*

*Proof.* By the definition of $r' \sim_\Gamma r$, $\mathsf{D}_\Gamma\varphi \in D_\Gamma(w^n) = D_\Gamma(v^n) \subseteq v^n$, whence $\varphi \in v^n$ follows by $\mathbf{T}_\mathsf{D}$.

**Lemma 9 ($\langle\!\langle . \mid . \rangle\!\rangle\circ$).** *Let $w \in W$, $\Psi \subset \mathbf{L}$ and $\langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \bigwedge\Psi' \in w$ for all finite $\Psi' \subseteq \Psi$. Let $a'_\Gamma = \langle\langle\{p_{i,\langle\!\langle\Gamma|\Delta\rangle\!\rangle\circ\Psi'} \Rightarrow \bigwedge\Psi' : \Psi' \subseteq_{fin} \Psi\}, \{i\}, \emptyset\rangle : i \in \Gamma\rangle$ Then $\Psi \subseteq t(w, a'_\Gamma \cup a_{w,\Delta} \cup a''_{\Sigma_e\setminus(\Gamma\cup\Delta)})$, where $a_{w,\Delta}$ is the restriction of $a_w$ as introduced in Definition 19, for all $a''_{\Sigma_e\setminus(\Gamma\cup\Delta)} \in Act_{\Sigma_e\setminus(\Gamma\cup\Delta)}$.*

*Proof.* Obviously $\Gamma$ is effective in $a'_\Gamma \cup a_{w,\Delta} \cup a''_{\Sigma\setminus(\Gamma\cup\Delta)}$ wrt $w$.

**Lemma 10 ($[\![ . \mid . ]\!]\circ$).** *Let $n < \omega$, $r = w^0a^0 \ldots a^{n-1}w^n \in R^n(IS)$, $\Gamma, \Delta \subseteq \Sigma$, $\Gamma \cap \Delta = \emptyset$. Let $s = s^0, \ldots, s^n \in (S(\Sigma_e))^{n+1}$ be consistent with $r$. Let $g_\Gamma \in Act_\Gamma$ be such that $\psi \in t(v^n, g_\Gamma \cup s'^n(v^n)_\Delta \cup g'_{\Sigma_e\setminus(\Gamma\cup\Delta)})$ for all $r' = v^0b^0 \ldots b^{n-1}v^n \in R^n(IS)$ such that $r' \sim_{\Gamma\cup\Delta} r$, all $s' \in (S(\Sigma_e))^{n+1}$ which are consistent with $r'$ and satisfy $s' \sim_{\Gamma\cup\Delta} s$, and all $g'_{\Sigma_e\setminus(\Gamma\cup\Delta)} \in Act_{\Sigma_e\setminus(\Gamma\cup\Delta)}$. Then $\langle\!\langle \Gamma \mid \Delta \rangle\!\rangle \circ \psi \in w^n$.*

*Proof.* Consider an $r'$ as in the lemma. Let $g'_i = \langle\emptyset, \{i\}, \emptyset\rangle$ for $i \in \Sigma\setminus(\Gamma\cup\Delta)$, Let $a = g_\Gamma \cup s'^n(v^n)_\Delta \cup g'_{\Sigma\setminus(\Gamma\cup\Delta)} \cup g'_e$. Let $\Phi$ be the union of all the sets of formulas in $\overline{a}_{w^n}$, i.e., the actions of the coalitions which are effective in $a$ wrt $w^n$ and, consequently, wrt any $v^n$ which is $\Gamma \cup \Delta$-similar to $w^n$. Then, by the definition of $t$, $t(v^n, a) = (\Phi \cup \Theta_{v^n}) + g'_e$. Assume that the least element of $g'_e$ is $\neg\psi$. Then, since $\psi \in t(v^n, a)$ and by the construction of $(\Phi \cup \Theta_{v^n}) + g'_e$, $\Phi \cup \Theta_{v^n} \vdash_{MP} \psi$. By compactness, there exists a finite $\Theta'_{v^n} \subset \Theta_{v^n}$ such that $\Phi \vdash_{MP} \bigwedge\Theta_{v^n} \Rightarrow \psi$. Let $\Theta'_{v^n} = \{\ominus\theta_1, \ldots, \ominus\theta_k\}$. Then $\bigwedge\Theta'_{v^n}$ is equivalent to $\ominus(\theta_1 \wedge \ldots \wedge \theta_k)$. The latter formula is in $\Theta_{v^n}$. Hence, for every $v^n$ which is the last state of an $r'$ as described in the lemma, there exists a $\xi \in \mathbf{L}$ such that $\ominus\xi \in \Theta_v^n$ and $\Phi \vdash_{MP} \ominus\xi \Rightarrow \psi$. Let

$$\Xi^n = \{\xi \in \mathbf{L} : \Phi \vdash_{MP} \ominus\xi \Rightarrow \psi,\ \ominus\xi \in \Theta_{v^n} \text{ for some } v^n \in W \text{ such that there exists an } r' \in R(IS), r' \sim_{\Gamma\cup\Delta} r \text{ with } v^n \text{ as its last state}\}.$$

A direct check shows that $\Xi^n$ is closed under disjunction. Assume that $\{\neg\xi : \xi \in \Xi^n\} \cup D_{\Gamma\cup\Delta}(w^n)$ is consistent for the sake of contradiction. Then, by Lemma 5, $D_{\Gamma\cup\Delta}(w^n) \vdash_{MP} \mathsf{P}_{\Gamma\cup\Delta}\neg\xi$ for every $\xi \in \Xi^n$.

Since $\Xi^n$ is closed under disjunction, we can conclude that $D_{\Gamma\cup\Delta}(w^n) \vdash_{MP} \mathsf{P}_{\Gamma\cup\Delta} \bigwedge\limits_{\xi\in\Xi^{n\prime}} \neg\xi$ for any finite $\Xi^{n\prime} \subset \Xi^n$. By Lemma 7, this entails that there exist an $r' \sim_{\Gamma\cup\Delta} r$ and an $s' \sim_{\Gamma\cup\Delta} s$ such that $s'$ is consistent with $r'$ and $\{\neg\xi : \xi \in \Xi^n\} \subseteq v^n$ where $v^n$ is the last state of $r'$. This contradicts the consistency of $v^n$ since, as we established above, for every $v^n$ with the specified properties, there exists a $\xi \in \Xi^n$ such that $\xi \in v^n$ as well. Hence $\{\neg\xi : \xi \in \Xi^{n\prime}\} \cup D_{\Gamma\cup\Delta}(w^n)$ is inconsistent. By compactness, this entails that there exists $D_{\Gamma\cup\Delta}(w^n) \vdash_{MP} \bigvee \Xi^{n\prime}$ for some finite $\Xi^{n\prime} \subset \Xi^n$. Now, having in mind that $\Phi \vdash_{MP} \ominus\xi \Rightarrow \psi$ for each of the $\xi$s in $\Xi^{n\prime}$, we can conclude that $\Phi \cup \{\ominus\delta : \delta \in D_{\Gamma\cup\Delta}(w^n)\} \vdash_{MP} \psi$. Hence there exist some finite $\Phi' \subseteq \Phi$ and $D' \subset D_{\Gamma\cup\Delta}(w^n)$ such that $\vdash \bigwedge \Phi' \wedge \ominus \bigwedge D' \Rightarrow \psi$. By the monotonicity of $\langle\langle\Gamma \mid \Delta\rangle\rangle\circ$, $\vdash \langle\langle\Gamma \mid \Delta\rangle\rangle\circ(\bigwedge\Phi' \wedge \ominus\bigwedge D') \Rightarrow \langle\langle\Gamma \mid \Delta\rangle\rangle\circ\psi$. Now, by Axiom $\langle\langle . \mid . \rangle\rangle\circ\ominus$, the latter entails $\vdash \langle\langle\Gamma \mid \Delta\rangle\rangle\circ(\bigwedge\Phi') \wedge \mathsf{D}_{\Gamma\cup\Delta}\bigwedge D' \Rightarrow \langle\langle\Gamma \mid \Delta\rangle\rangle\circ\psi$. For any $v^n$ as in the lemma, the definition of $\Phi$ entails $\langle\langle\Gamma \mid \Delta\rangle\rangle \circ (\bigwedge\Phi') \in v^n$, and $D' \subset D_{\Gamma\cup\Delta}(w^n) = D_{\Gamma\cup\Delta}(v^n)$ entails $\bigwedge D' \in v^n$. Hence, finally, $\langle\langle\Gamma \mid \Delta\rangle\rangle \circ \psi \in v^n$ as well.

**Lemma 11 ($\ominus$ and $(.\mathsf{S}.)$).** *Let $n$, $r \in R^n(IS)$ and $s \in (S(\Sigma_e))^{n+1}$ be as in the previous lemmata. Then, $\circ^n\mathsf{I} \in w^n$ and, for any two formulas $\varphi, \psi \in \mathbf{L}$, $\ominus\varphi \in w^n$ iff $n > 0$ and $\varphi \in w^{n-1}$, and $(\varphi\mathsf{S}\psi) \in w^n$ iff there exists a $k \leq n$ such that $\psi \in w^k$ and $\varphi \in w^{k+1}, \ldots, w^n$.*

We omit the proof of this lemma as it contains nothing specific to $ATL_{iR}^{DPC}$. We only note that the fact $\circ^n\mathsf{I} \in w^n$ guarantees that the presence of $(.\mathsf{S}.)$ does not affect the compactness of the system, despite that $(.\mathsf{S}.)$ is an iterative operator.

**Theorem 1 (truth lemma).** *Let $n < \omega$, $r \in R^n(IS)$, $r = w^0a^0 \ldots a^{n-1}w^n$, and let $s \in S(\Sigma_e)^{n+1}$ be consistent with $r$. Then $IS, r \models \varphi$ iff $\varphi \in w^n$.*

*Proof.* Induction on the construction of $\varphi$. The cases of $\varphi$ being $\bot$, an atomic proposition, or an implication, are trivial, and we skip them. For $\varphi$ of the forms $\mathsf{D}_\Gamma\psi$, $\langle\langle\Gamma \mid \Delta\rangle\rangle \circ \psi$ and $\ominus\psi$ and $(\psi\mathsf{S}\chi)$ the theorem follows from Lemmata 8 and 7, 9 and 10 and Lemma 11, respectively.

**Corollary 1 (strong completeness for basic $ATL_{iR}^{DPC}$).** *Let $x$ be a consistent subset of $\mathbf{L}$ and let $\mathsf{I} \in x$. Then there exists an initial state $l \in I$ and a vector of strategies $s \in S(\Sigma_e)$ such that $IS, s, l \models \varphi$ for all $\varphi \in x$.*

*Proof.* Let $w$ be any MCS such that $x \subseteq w$. Then Theorem 1 entails that $IS, s_{IS}, l_w \models \varphi$ for all $\varphi \in w$.

## Concluding Remarks and Future Work

We have shown that extending the game-theoretic operator ⟨⟨.⟩⟩ of $ATL_{iR}^{DP}$ by a second coalition parameter to denote players who act according to the strategic context, we obtain a language for epistemic *ATL* with strategic contexts which admits an axiom system that is a straightforward revision of the system for $ATL_{iR}^{DP}$ from our work [9]. So far we have established the completeness of that system for a subset of $ATL_{iR}^{DP}$ with strategy contexts which lacks the combinations of ⟨⟨.⟩⟩ with the iterative future temporal operator (.U.) and its derivatives $\Diamond$ and $\Box$. Taking advantage of the compactness of this subset, we have obtained a strong completeness theorem. We have established the semantical compatibility between our proposed system and the systems of *ATL* with strategy contexts and complete information from the literature, especially [5,15]. We intend to further investigate the axiomatizability of $ATL_{iR}^{DPC}$, seeking to establish weak completeness theorems and the decidability of validity of bigger subsets.

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## References

1. Ågotnes, T., Goranko, V., Jamroga, W.: Strategic Commitment and Release in Logics for Multi-Agent Systems (Extended Abstract). Technical report IfI-08-01, TU Clausthal (May 2008); Ågotnes, T., Goranko, V., Jamroga, W.: Strategic Commitment and Release in Logics for Multi-Agent Systems (Extended Abstract). In: Bonanno, G., Löwe, B., van der Hoek, W. (eds.) Logic and the Foundations of Game and Decision Theory. LNCS, vol. 6006, Springer, Heidelberg (2010)
2. Alur, R., Henzinger, T., Kupferman, O.: Alternating-time Temporal Logic. In: Proceedings of FCS 1997, pp. 100–109 (1997)
3. Alur, R., Henzinger, T., Kupferman, O.: Alternating-time temporal logic. Journal of the ACM 49(5), 1–42 (2002)
4. Asarin, E.: EQINOCS Research Project ANR 11 BS02 004 03, `http://www.liafa.univ-paris-diderot.fr/~eqinocs/`
5. Brihaye, T., Da Costa, A., Laroussinie, F., Markey, N.: ATL with Strategy Contexts and Bounded Memory. In: Artemov, S., Nerode, A. (eds.) LFCS 2009. LNCS, vol. 5407, pp. 92–106. Springer, Heidelberg (2008)
6. Chatterjee, K., Henzinger, T.A., Piterman, N.: Strategy logic. Information and Computation 208(6), 677–693 (2010)
7. Fagin, R., Halpern, J., Moses, Y., Vardi, M.: Reasoning about Knowledge. MIT Press (1995)
8. Goranko, V., van Drimmelen, G.: Decidability and Complete Axiomatization of the Alternating-time Temporal Logic. Theoretical Computer Science 353(1-3), 93–117 (2006)
9. Guelev, D.P., Dima, C., Enea, C.: An Alternating-time Temporal Logic with Knowledge, Perfect Recall and Past: Axiomatisation and Model-checking. Journal of Applied Non-Classical Logics 21(1), 93–131 (2011)

10. Halpern, J., van der Meyden, R., Vardi, M.: Complete Axiomatizations for Reasoning about Knowledge and Time. SIAM Journal on Computing 33(3), 674–703 (2004)
11. Jamroga, W., van der Hoek, W.: Agents That Know How to Play. Fundamenta Informaticae 63(2-3), 185–219 (2004)
12. Lichtenstein, O., Pnueli, A.: Propositional temporal logics: decidability and completeness. Logic Journal of the IGPL 8(1), 55–85 (2000)
13. Lomuscio, A., Raimondi, F.: Model checking knowledge, strategies, and games in multi-agent systems. In: Proceedings of AAMAS 2006, pp. 161–168. ACM Press (2006)
14. Lomuscio, A., Raimondi, F.: The Complexity of Model Checking Concurrent Programs Against CTLK Specifications. In: Baldoni, M., Endriss, U. (eds.) DALT 2006. LNCS (LNAI), vol. 4327, pp. 29–42. Springer, Heidelberg (2006)
15. Lopes, A.D.C., Laroussinie, F., Markey, N.: ATL with strategy contexts: Expressiveness and model checking. In: FSTTCS. LIPIcs, vol. 8, pp. 120–132 (2010)
16. Pauly, M.: A modal logic for coalitional power in games. Journal of Logic and Computation 12(1), 149–166 (2002)
17. Schobbens, P.Y.: Alternating-time logic with imperfect recall. ENTCS 85(2), 82–93 (2004); Proceedings of LCMAS 2003
18. van der Hoek, W., Wooldridge, M.: Cooperation, Knowledge and Time: Alternating-time Temporal Epistemic Logic and Its Applications. Studia Logica 75, 125–157 (2003)
19. van der Meyden, R., Wong, K.-S.: Complete Axiomatizations for Reasoning about Knowledge and Branching Time. Studia Logica 75(1), 93–123 (2003)
20. Walther, D., van der Hoek, W., Wooldridge, M.: Alternating-time Temporal Logic with Explicit Strategies. In: Samet, D. (ed.) TARK, pp. 269–278. ACM Press (2007)
21. Wang, F., Huang, C.-H., Yu, F.: A Temporal Logic for the Interaction of Strategies. In: Katoen, J.-P., König, B. (eds.) CONCUR 2011 – Concurrency Theory. LNCS, vol. 6901, pp. 466–481. Springer, Heidelberg (2011)